Continue

Given $a_{i} = (a_{i1} \dots a_{in}) \in \mathbf R^{n}$, $i = 1 \dots m$, and $b = (b_{1} \dots b_{n}) \in \mathbf R^{n}$; then for any $\epsilon > 0$ there exist integers q_{i}, $i = 1 \dots m$, and p_{j}, $j = 1 \dots n$, such that $$\left| \sum_{i=1}^{m} q_{i} a_{ij} - p_{j} - b_{j} \right| < \epsilon ,\ \ 1 \leq j \leq n,$$ if and only if for any $r_{1} \dots r_{n} \in \mathbf Z$ such that $$\sum_{j=1}^{n} a_{ij} r_{j} \in \mathbf Z ,\ \ i = 1 \dots m,$$ the number $$\sum_{j=1}^{n} b_{j} r_{j}$$ is also an integer. This theorem was first proved in 1884 by L. Kronecker (see [1]). Kronecker's theorem is a special case of the following theorem [2], which describes the closure of the subgroup of the torus $T^{n} = \mathbf R^{n} / \mathbf Z^{n}$ generated by the elements $a_{i} + \mathbf Z^{n}$, $i = 1 \dots m$: The closure is precisely the set of all classes $b + \mathbf Z^{n}$ such that, for any numbers $r_{1} \dots r_{n} \in \mathbf Z$ with $$\sum_{j=1}^{n} a_{ij} r_{j} \in \mathbf Z ,\ \ i = 1 \dots n,$$ one has also $$\sum_{j=1}^{n} b_{j} r_{j} \in \mathbf Z .$$ (Cf. [2].) Under the assumptions of Kronecker's theorem, this closure is simply T^{n}. This means that the subgroup of all elements of the form $$\sum_{i=1}^{m} q_{i} (a_{i} + \mathbf Z)^{n} ,$$ where $q_{i} \in \mathbf Z$, is dense in T^{n}, while the subgroup of vectors $$\sum_{i=1}^{m} q_{i} a_{i} + p,$$ where $p \in \mathbf Z^{n}$, is dense in $\mathbf R^{n}$. Kronecker's theorem can be derived from the duality theory for commutative topological groups (cf. Topological group), [3]. In the case $m = 1$, Kronecker's theorem becomes the following proposition: A class $\omega + \mathbf Z^{n}$, where $\omega = (\omega_{1} \dots \omega_{n}) \in \mathbf R^{n}$, generates T^{n} as a topological group if and only if the numbers $1, \omega_{1} \dots \omega_{n}$ are linearly independent over the field $\mathbf Q$ of rational numbers. In particular, the torus T^{n} as a topological group is monothetic, i.e. is generated by a single element. References [1] L. Kronecker, "Näherungsweise ganzzahlige Auflösung linearer Gleichungen" , Werke , 3 , Chelsea, reprint (1968) pp. 47–109[2] N. Bourbaki, "Elements of mathematics. General topology" , Addison-Wesley (1966) (Translated from French)[3] L.S. Pontryagin, "Topological groups" , Princeton Univ. Press (1958) (Translated from Russian) The last statement above can be rephrased as: If $\omega_{1} \dots \omega_{n}$ are linearly independent over $\mathbf Q$, then the set $B = \{ (\{ k \omega_{1} \} \dots \{ k \omega_{n} \}) : k \in \mathbf Z \}$ is dense in $(0, 1)$. Here $\{ x \} = x - [x]$ denotes the fractional part of x (cf. Fractional part of a number). In fact, the set B is even uniformly distributed, cf. Uniform distribution. References [a1] G.H. Hardy, E.M. Wright, "An introduction to the theory of numbers" , Oxford Univ. Press (1979) pp. Chapt. 23[a2] J.W.S. Cassels, "An introduction to diophantine approximation" , Cambridge Univ. Press (1957) How to Cite This Entry: Kronecker theorem. Encyclopedia of Mathematics. URL: Theorem about Diophantine approximations For the theorem about the real analytic Eisenstein series, see Kronecker limit formula. In mathematics, Kronecker's theorem is a theorem about diophantine approximation, introduced by Leopold Kronecker (1884). Kronecker's approximation theorem had been firstly proved by L. Kronecker in the end of the 19th century. It has been now revealed to relate to the idea of n-torus and Mahler measure since the later half of the 20th century. In terms of physical systems, it has the consequence that planets in circular orbits moving uniformly around a star will, over time, assume all alignments, unless there is an exact dependency between their orbital periods. Kronecker's theorem is a result about Diophantine approximations that generalizes Dirichlet's approximation theorem to multiple variables. The Kronecker approximation theorem is classically formulated as follows. Given real n-tuples α i = (α i 1 , … , α i n) ∈ R n , i = 1 , … , m {\displaystyle \alpha _{i}=(\alpha _{i1},\dots ,\alpha _{in})\in \mathbb {R} ^{n},i=1,\dots ,m} and β = (β 1 , … , β n) ∈ R n {\displaystyle \beta =(\beta _{1},\dots ,\beta _{n})\in \mathbb {R} ^{n}} , the condition: ∀ ϵ > 0 ∃ q i , p j ∈ Z : | ∑ i = 1 m q i α i j − p j − β j | < ϵ , 1 ≤ j ≤ n {\displaystyle \forall \epsilon >0\,\exists q_{i},p_{j}\in \mathbb {Z} :{\biggl |}\sum _{i=1}^{m}q_{i}\alpha _{ij}-p_{j}-\beta _{j}{\biggr |}